

ON A SOLUTION OF LAPLACE'S EQUATION WITH AN APPLICATION TO THE TORSION PROBLEM FOR A POLYGON WITH REENTRANT ANGLES†

BY

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I. INTRODUCTION

In this paper will be found a general method of solution of the two-dimensional Laplace equation with certain boundary conditions prescribed along the sides of any rectilinear polygon. The applicability of this method to the solution of technical problems will be illustrated by the treatment of the torsion problem for an infinite T-section. The mathematical solution of this problem, and of that for a finite T, is not to be found in the existing literature.‡

It will be seen that this scheme of solving Laplace's equation can readily be applied to any region which can be mapped conformally onto the upper half-plane in such a way that the boundary of the region goes into the entire real axis while the interior of the region transforms into the upper half-plane.

Despite the considerable scientific interest in the behavior of structural members subjected to pure torsion, only a limited number of torsion problems have been brought within the range of mathematical analysis. A torsion problem is solved when one has determined a function Φ which satisfies the equation $\nabla^2\Phi=0$, and which on the boundary of the section subjected to torsion reduces to § $\Phi^* = \frac{1}{2}(x^2 + y^2)$. The determination of Φ for such simple regular sections as the circle, ellipse, equilateral triangle, and rectangle has been achieved with comparative ease.||

In 1908, F. Kötter¶ succeeded in obtaining a solution of the torsion problem for an L-section by the use of the known solution for the rectangle and by application of the scheme of conformal transformation. Kötter's

† Presented to the Society, April 18, 1930; received by the editors February 24, 1931.

‡ For practical methods of solution, and for an extensive bibliography, see 15th Annual Report of the National Advisory Committee for Aeronautics, 1929, pp. 675-719, *The torsion of members having sections common in aircraft construction*, by W. Trayer and H. W. March. This work was also published by the Bureau of Aeronautics, Navy Department, as a separate Report No. 334, bearing the same title (U. S. Government Printing Office, 1930).

§ Love, A. E. H., *Theory of Elasticity*, pp. 318-319, 3d edition, 1920.

|| Love, loc. cit., pp. 322-333.

¶ Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften, 1908, pp. 935-955.

method, however, does not lend itself readily to the solution of the problem involving more than one reentrant angle. Apparently the first paper proposing a general method of solution of the two-dimensional Laplace equation, with the torsion boundary conditions prescribed along the sides of a rectilinear polygon, was published in 1921 by E. Trefftz.[†] The method which Trefftz employed possesses a distinct disadvantage in that it makes the ultimate solution of the problem (as applied to an L-section) depend upon some graphical scheme. Moreover, the success of the Trefftz method depends upon the particular form of the boundary conditions occurring in the torsion problem. The method used in the present paper is more general, and does not restrict the choice of the boundary conditions to those of the torsion problem.

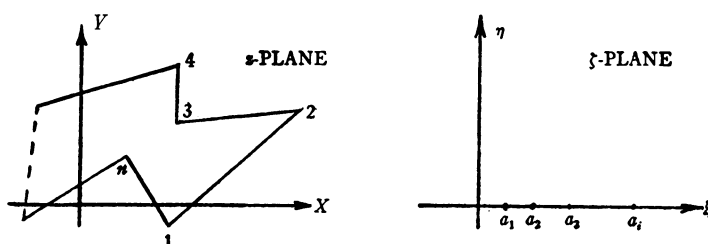


FIG. 1

II. BOUNDARY VALUE PROBLEM

It is known from a fundamental theorem of potential theory that a harmonic function is uniquely determined by the values assigned along the boundary of the region within which the harmonic function is sought, the boundary conditions and the region being subject to certain well known assumptions of continuity, connectivity, etc. In particular, when the values of the potential function are prescribed along the ξ -axis, the value of the function at any point of the upper half-plane is given by

$$(1) \quad \Phi(\rho, \alpha) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\Phi^*(\xi) \rho \sin \alpha d\xi}{\rho^2 - 2\xi\rho \cos \alpha + \xi^2}$$

where ξ is the value of ζ along the real axis of the complex ζ -plane, (ρ, α) are the polar coordinates of the point in the ζ -plane, and $\Phi^*(\xi)$ is the function prescribed on the boundary.

Consider a rectilinear polygon in the complex z -plane (Fig. 1), and denote the interior angles of the polygon at the points 1, 2, 3, \dots , n by $\pi\alpha_1, \pi\alpha_2, \pi\alpha_3, \dots, \pi\alpha_n$. It is allowable for some of the angular points of the polygon to recede to infinity. It is possible to map the z -plane conformally on

[†] Mathematische Annalen, vol. 82 (1921), pp. 97-112.

the ζ -plane in such a way that the boundary of the polygon transforms into the real axis of the ζ -plane, and the interior of the polygon maps into the upper half of the ζ -plane. Such a transformation was given independently by Schwartz and Christoffel† and its form is

$$(2) \quad z = C_1 \int \prod_{i=1}^n (\zeta - a_i)^{\alpha_i - 1} d\zeta + C_2,$$

where C_1 and C_2 are constants determinable from the orientation, scale, and position of the polygon. The points $a_1, a_2, a_3, \dots, a_n$ are the points on the real axis of the ζ -plane corresponding to the angular points $(1, 2, 3, \dots, n)$ of the polygon. Since it is possible to transform any three given points of the z -plane into any three desired points of the ζ -plane, we are free to prescribe the location of any three points a_i along the axis of reals, while the position of the remaining $(n-3)$ points will be determined from the dimensions of the polygon. It is necessary to remark that the order of the points a_1, a_2, \dots, a_n along the ξ -axis must be the same as that of the angular points $(1, 2, \dots, n)$ around the polygon.

If one succeeds in integrating (2), and further if one decomposes it into its real and imaginary parts, there result two equations of transformation

$$(3) \quad \begin{aligned} x &= g_1(\xi, \eta), \\ y &= g_2(\xi, \eta), \end{aligned}$$

where $g_1(\xi, \eta)$ and $g_2(\xi, \eta)$ are real functions of the real variables ξ and η . These equations of transformation, when applied to the analytic equations representing the boundary of the polygon in the z -plane, will transform it into the real axis of the ζ -plane.

Consider now the problem of determining the harmonic function $\Phi(x, y)$ in the interior of the region bounded by the rectilinear polygon, and let the prescribed values of $\Phi(x, y)$ along the sides of this polygon be given by $\Phi^* = f(x, y)$. The application of the equations of transformation (3) gives

$$(4) \quad \Phi^* = f[g_1(\xi, \eta), g_2(\xi, \eta)].$$

In (4), Φ^* is a function of ξ only, since, by hypothesis, the boundary of the polygon in the z -plane is transformed into the ξ -axis of the ζ -plane.

The substitution of this new boundary value function in (1) gives an expression for the determination of the value of Φ at any point (ξ, η) of the upper half of the ζ -plane. In view of the fact that the upper half of the ζ -plane corresponds to the interior of the polygon, it is clear that the integral (1),

† Schwartz, *Gesammelte Werke*, vol. II, pp. 65-83. Christoffel, *Annali di Matematica Pura ed Applicata*, (2), vol. 1 (1867), pp. 95-103. *Ibid.*, vol. 4 (1871), pp. 1-9.

together with the equations of transformation (3), constitute the solution of the problem in parametric form.

III. APPLICATION TO THE TORSION PROBLEM

The foregoing considerations can be applied to the solution of the torsion problem for a long prism whose cross section is in the shape of the letter T.

In this case we wish to determine a function Φ which satisfies the differential equation

$$(5) \quad \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

and assumes on the boundary of the section the values given by

$$(6) \quad \Phi^* = \frac{x^2 + y^2}{2}.$$

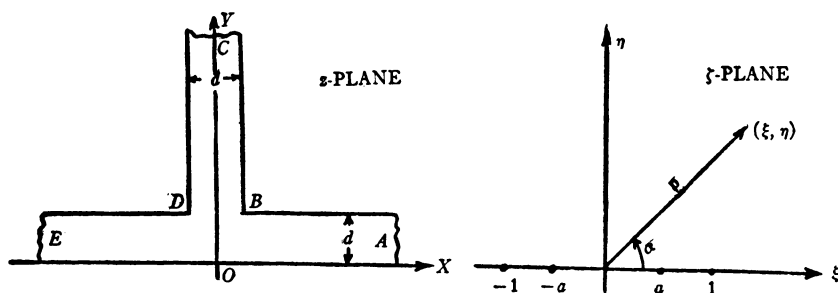


FIG. 2

Let the width of the flange and web be d . We shall consider the web and flange of the T as extending indefinitely along the X and Y axes. This assumption will lead to useful results since the behavior of Φ , at points of the flange and web sufficiently far removed from the reentrant angles, is essentially the same as that in a rectangle.†

The theory outlined in §II is applicable to the section in question. We are at liberty to transform any three desired points of the boundary of the polygon in the z -plane into any three points of the ξ -axis. Let the point B (see Fig. 2) go into the point 1 of the ξ -axis, the point $C(i\infty)$ into ∞ of the ξ -axis, and the point A into some point a of the ξ -axis. The value of a will be determined from the dimensions of the polygon. It is obvious from symmetry that the point D will go into -1 , and the point E into $-a$.

† For an exposition of the torsion theory see Love, loc. cit., pp. 315-333; Trayer and March, loc. cit., pp. 10-20.

A direct substitution of the coördinates of the points a_i and of the values of the interior angles $\alpha_i\pi$ into the Schwartz integral (2) gives

$$(7) \quad z = C_1 \int \frac{(\zeta^2 - 1)^{1/2}}{\zeta^2 - a^2} d\zeta + C_2.$$

It is to be remarked that the values of the interior angles of the T-polygon are 0, $(3/2)\pi$, 0, $(3/2)\pi$, and 0, at the points A , B , C , D , and E respectively.

The integral (7) can be readily evaluated by making an elementary transformation $\zeta = \sec \theta$ and dividing the numerator of the resulting expression by the denominator. The result is

$$(8) \quad z = C_1 \log (\zeta + (\zeta^2 - 1)^{1/2}) + \frac{C_1}{a} (1 - a^2)^{1/2} \tan^{-1} \frac{a(\zeta^2 - 1)^{1/2}}{(1 - a^2)^{1/2}\zeta} + C_2.$$

The constants of integration C_1 and C_2 are complex numbers, and must be determined from the geometry of the polygon. Choosing for this purpose the points 0, -1 , and 1 in the ζ -plane, and substituting the corresponding points of the z -plane in (8), one obtains with little effort the following values:

$$a = \frac{1}{5^{1/2}}, \quad C_1 = \frac{di}{\pi}, \quad C_2 = \frac{d}{2} + id,$$

where d is the width of the section. The substitution of these constants in (8) gives the explicit form of the transformation of the T-polygon into the upper half of the ζ -plane. It is

$$(9) \quad z = \frac{d}{\pi} i \log (\zeta + (\zeta^2 - 1)^{1/2}) - \frac{2d}{\pi} i \tan^{-1} \frac{(\zeta^2 - 1)^{1/2}}{2\zeta} + \frac{d}{2} + id.$$

We proceed next to decompose the equation of transformation (9) into its real and imaginary parts. By substitution of the symbols

$$A = \zeta + (\zeta^2 - 1)^{1/2} \quad \text{and} \quad B = \frac{(\zeta^2 - 1)^{1/2}}{2\zeta},$$

(9) becomes

$$z = \frac{2di}{\pi} \left(\frac{1}{2} \log A + \tan^{-1} B \right) + \frac{d}{2} + id.$$

Then

$$\begin{aligned} y = \frac{1}{2i}(z - \bar{z}) &= \frac{d}{\pi} \left(\frac{1}{2} \log A + \frac{1}{2} \log \bar{A} - \tan^{-1} B - \tan^{-1} \bar{B} \right) + \\ &= \frac{d}{\pi} \left(\frac{1}{2} \log A \bar{A} - \tan^{-1} \frac{B + \bar{B}}{1 - B\bar{B}} \right) + d \\ &= \frac{d}{\pi} \left(\log |A| - \tan^{-1} \frac{\Re(B)}{1 - |B|^2} \right) + d, \end{aligned}$$

and

$$\begin{aligned} x = \frac{1}{2}(z + \bar{z}) &= \frac{di}{\pi} \left(\frac{1}{2} \log \frac{A}{\bar{A}} - \tan^{-1} \frac{B - \bar{B}}{1 + B\bar{B}} \right) + \frac{d}{2} \\ &= \frac{di}{\pi} \left(i \arg A - \tan^{-1} \frac{2i\Im(B)}{1 + |B|^2} \right) + \frac{d}{2} \\ &= \frac{d}{\pi} \left(-\arg A - \tanh^{-1} \frac{2\Im(B)}{1 + |B|^2} \right) + \frac{d}{2}. \end{aligned}$$

In order to obtain the transformed boundary value function for substitution in (1), one must compute the value of $\Phi^* = \frac{1}{2}(x^2 + y^2)$ in terms of ξ . Since the boundary of the T-polygon transforms into the real axis of the ζ -plane, (6) will be a function of ξ only, and one obtains with a little effort the following equations:

$$(10) \quad \begin{aligned} x &= \pm \frac{d}{2} \begin{cases} + & \text{for } \xi > 1 \\ - & \text{for } \xi < -1, \end{cases} \\ y &= \frac{d}{\pi} \left(\cosh^{-1} |\xi| + 2 \tan^{-1} \frac{2|\xi|}{(\xi^2 - 1)^{1/2}} \right), \text{ for } |\xi| > 1. \end{aligned}$$

On account of the multiple-valued functions entering in (9), it is necessary to compute four sets of equations, analogous to (10), which correspond to the ranges† $(-1 < \xi < -1/5^{1/2})$, $(-1/5^{1/2} < \xi < 0)$, $(0 < \xi < 1/5^{1/2})$, and $(1/5^{1/2} < \xi < 1)$. Since the equations of transformation so obtained lead to six distinct forms, it is necessary to decompose the range of the integral (1) into six parts corresponding to the different forms of the boundary function defined over these ranges. A reference to (10) partly indicates the complexity of the resulting integrals.

It will be shown next that it is possible to dispense with the task of evaluating the four integral expressions corresponding to the range $-1 < \xi < 1$ by means of the following expedient. Consider the function

$$(11) \quad \Phi_1 = \frac{x^2 - y^2}{2} + yd$$

which obviously satisfies $\nabla^2 \Phi = 0$. Let Φ_2 be the function which satisfies $\nabla^2 \Phi = 0$, and which assumes on the boundary the value

$$(12) \quad \Phi_2^* = y^2 - yd.$$

† The appearance of $1/5^{1/2}$ is to be expected since, for $\zeta = 1/5^{1/2}$, (9) is not defined, inasmuch as in the period strip $\tan \zeta$ assumes every complex value except $\pm i$. The points $\zeta = \pm 1/5^{1/2}$ and $\zeta = \pm 1$ are the singular points of transformation (9).

It is clear that (12) vanishes when $y=0$, or when $y=d$, and that

$$\Phi_1 + \Phi_2^* = \frac{x^2 + y^2}{2} = \Phi^*.$$

Thus, if the function $\Phi_2(x, y)$ satisfying Laplace's equation with the boundary condition (12) be found, then the function

$$(13) \quad \Phi = \Phi_1 + \Phi_2$$

is determined. The advantage in seeking $\Phi_2(x, y)$ rather than $\Phi(x, y)$ directly from (1) lies in the fact that along the boundary of the flange (i.e. the portion corresponding to the ξ -axis between -1 and $+1$) $\Phi_2^* \equiv 0$. Consequently four of the six integral expressions vanish, since they involve the boundary value function in the numerator of the integrand.

The substitution of (10) in (12) gives for $|\xi| > 1$

$$(14) \quad \Phi_2^* = y^2 - yd = \frac{d^2}{\pi} \left(\cosh^{-1} |\xi| + 2 \tan^{-1} \frac{2|\xi|}{(\xi^2 - 1)^{1/2}} \right) - \frac{d^2}{\pi} \left(\cosh^{-1} |\xi| + 2 \tan^{-1} \frac{2\xi}{(\xi^2 - 1)^{1/2}} \right),$$

and for $-1 < \xi < 1$,

$$\Phi_2^* = 0.$$

Using (1), and observing that (14) is an even function, we have

$$(15) \quad \begin{aligned} \Phi_2(\rho, \alpha) &= \int_{-\infty}^1 \rho \frac{\sin \alpha}{\pi} [y^2(\xi) - y(\xi)d] (\rho^2 - 2\xi\rho \cos \alpha + \xi^2)^{-1} d\xi \\ &\quad + \int_1^{\infty} \rho \frac{\sin \alpha}{\pi} [y^2(\xi) - y(\xi)d] (\rho^2 - 2\xi\rho \cos \alpha + \xi^2)^{-1} d\xi \\ &= \frac{\rho}{\pi} \int_1^{\infty} [y^2(\xi) - y(\xi)d] \Omega(\rho, \xi) d\xi, \end{aligned}$$

where

$$(16) \quad \Omega(\rho, \xi) = \frac{\sin \alpha}{\rho^2 + 2\xi\rho \cos \alpha + \xi^2} + \frac{\sin \alpha}{\rho^2 - 2\xi\rho \cos \alpha + \xi^2}.$$

IV. EVALUATION OF THE INTEGRALS

A brief perusal of the integrals involved in (15) is sufficient to make one abandon the hope of evaluating them in closed form. For reasons which will be made clear later, it is found advantageous to divide the range of integration from 1 to ∞ into two ranges, say from 1 to m and from m to ∞ , where m is some positive number greater than 1. As will be seen, the choice of m will

depend upon the degree of accuracy desired in computing the value of $\Phi_2(\rho, \alpha)$.

Since

$$\frac{\sin \alpha}{1 - 2 \frac{\rho}{\xi} \cos \alpha + \frac{\rho^2}{\xi^2}} = \sum_{n=1}^{\infty} \left(\frac{\rho}{\xi} \right)^{n-1} \sin n\alpha, \quad \text{for } \left| \frac{\rho}{\xi} \right| < 1,$$

(16) can be written as

$$(16') \quad \Omega(\rho, \xi) = \begin{cases} \frac{2}{\xi^2} \sum_{n=0}^{\infty} \left(\frac{\rho}{\xi} \right)^{2n} \sin (2n+1)\alpha, & \text{if } \left| \frac{\rho}{\xi} \right| < 1, \\ \frac{2}{\rho^2} \sum_{n=0}^{\infty} \left(\frac{\xi}{\rho} \right)^{2n} \sin (2n+1)\alpha, & \text{if } \left| \frac{\xi}{\rho} \right| < 1. \end{cases}$$

Moreover, for some value of $\xi \geq m$, (14) may be simplified, inasmuch as

$$\cosh^{-1} \xi = \log (\xi + (\xi^2 - 1)^{1/2}),$$

which is asymptotically equal to $\log 2\xi$, and

$$\tan^{-1} \frac{2\xi}{(\xi^2 - 1)^{1/2}} \sim \tan^{-1} 2.$$

With these approximations (14) reads

$$(17) \quad \Phi_2^* = y^2 - yd = d^2(a_1 \log^2 2\xi + a_2 \log^2 \xi + a_3) + O\left(\frac{1}{\xi^2}\right),$$

where

$$a_1 = \frac{1}{\pi^2}, \quad a_2 = \frac{4 \tan^{-1} 2}{\pi^2} - \frac{1}{\pi}, \quad a_3 = \frac{4}{\pi^2} (\tan^{-1} 2)^2 - \frac{2}{\pi} \tan^{-1} 2.$$

Now define

$$(18) \quad \begin{aligned} \Theta(\gamma, \delta) &\equiv \int_{\gamma}^{\delta} (y^2 - yd) \Omega(\rho, \xi) d\xi \\ &= \begin{cases} 2 \sum_{n=0}^{\infty} \rho^{2n} \sin (2n+1)\alpha \int_{\gamma}^{\delta} (y^2 - yd) \xi^{-2n-2} d\xi, & \text{if } \rho < \gamma, \\ 2 \sum_{n=0}^{\infty} \frac{\sin (2n+1)\alpha}{\rho^{2n+2}} \int_{\gamma}^{\delta} (y^2 - yd) \xi^{2n} d\xi, & \text{if } \rho > \delta. \end{cases} \end{aligned}$$

Conditions for inversion of the order of integration and summation are clearly satisfied, if in (16) ρ is prevented from approaching ξ by an arbitrarily

small positive quantity ϵ . Moreover $\Theta(\gamma, \delta)$ is continuous in both γ and δ , when $\gamma \neq 0$, $\delta \neq 0$.

Now, if $\rho > m$,

$$\begin{aligned} \Phi_2(\rho, \alpha) &= \frac{\rho}{\pi} \left\{ \Theta(1, m) + \lim_{b \rightarrow \infty} \lim_{\epsilon \rightarrow 0} [\Theta(m, \rho - \epsilon) + \Theta(\rho + \epsilon, b)] \right\} \\ (19) \quad &= \frac{\rho}{\pi} \left\{ \Theta(1, m) + \Theta(m, \rho) + \lim_{b \rightarrow \infty} \Theta(\rho, b) \right\}, \end{aligned}$$

since Θ is continuous.

Substituting in (19) from (18) and simplifying we have

$$\begin{aligned} \Phi_2(\rho, \alpha) - \frac{\rho}{\pi} \Theta(1, m) &= \frac{2d^2}{\pi} \left\{ \frac{\pi a_1}{2} \log^2 2\rho + \frac{\pi a_2}{2} \log 2\rho + \frac{\pi a_3}{2} + \frac{\pi^2 \alpha - \pi \alpha^2}{2} a_1 \right. \\ (20) \quad &\quad - \frac{a_3 + a_2 \log 2m + a_1 \log^2 2m}{2} \tan^{-1} \frac{2m\rho \sin \alpha}{\rho^2 - m^2} \\ &\quad + (2a_1 \log 2m + a_2) \sum_{n=0}^{\infty} \left(\frac{m}{\rho} \right)^{2n+1} \frac{\sin (2n+1)\alpha}{(2n+1)^2} \\ &\quad \left. - 2a_1 \sum_{n=0}^{\infty} \left(\frac{m}{\rho} \right)^{2n+1} \frac{\sin (2n+1)\alpha}{(2n+1)^3} \right\}. \end{aligned}$$

In computing (20) use was made of the equalities

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \sin (2n+1)\alpha = \frac{1}{2} \tan^{-1} \frac{2x \sin \alpha}{1 - x^2}$$

and

$$\sum_{n=0}^{\infty} \frac{\sin (2n+1)\alpha}{(2n+1)^3} = \frac{\pi^2 \alpha - \pi \alpha^2}{8}.$$

In like manner for $\rho < m$ we obtain

$$\begin{aligned} \Phi_2(\rho, \alpha) - \frac{\rho}{\pi} \Theta(1, m) &= \frac{2d^2}{\pi} \left\{ \frac{a_1 \log^2 2m + a_2 \log 2m + a_3}{2} \tan^{-1} \frac{2m\rho \sin \alpha}{m^2 - \rho^2} \right. \\ (21) \quad &\quad + (2a_1 \log 2m + a_2) \sum_{n=0}^{\infty} \left(\frac{\rho}{m} \right)^{2n+1} \frac{\sin (2n+1)\alpha}{(2n+1)^2} \\ &\quad \left. + 2a_1 \sum_{n=0}^{\infty} \left(\frac{\rho}{m} \right)^{2n+1} \frac{\sin (2n+1)\alpha}{(2n+1)^3} \right\}. \end{aligned}$$

In order to complete the solution of the problem it remains to establish the magnitude of m and to evaluate $\Theta(1, m)$. It will be shown next that one

attains a sufficiently high degree of accuracy by choosing $m = 1$, and noting that† $\Theta(1, 1) = 0$.

The expression for the relative error made in assuming

is

$$\int_1^\infty \frac{\log 2\xi + 2 \tan^{-1} 2}{\xi^2 \pm 2\xi\rho \cos \alpha + \rho^2} d\xi \rightarrow \int_1^\infty \frac{\cosh^{-1} \xi + 2 \tan^{-1} \frac{2\xi}{(\xi^2 - 1)^{1/2}}}{\xi^2 \pm 2\xi\rho \cos \alpha + \rho^2} d\xi$$

$$E = \frac{\int_1^\infty [f(\xi) - \phi(\xi)] d\xi}{\int_1^\infty f(\xi) d\xi},$$

where

$$f(\xi) = \frac{\cosh^{-1} \xi + 2 \tan^{-1} \frac{2\xi}{(\xi^2 - 1)^{1/2}}}{\xi^2 \pm 2\xi\rho \cos \alpha + \rho^2}, \quad \phi(\xi) = \frac{\log 2\xi + 2 \tan^{-1} 2}{\xi^2 \pm 2\xi\rho \cos \alpha + \rho^2},$$

and for m sufficiently large

$$(22) \quad E \rightarrow \frac{\int_1^m [f(\xi) - \phi(\xi)] d\xi}{\int_1^\infty f(\xi) d\xi}.$$

It can be readily established that the numerators of the integrands in the foregoing expressions for E are monotone increasing functions and that their difference is a monotone decreasing function within the limits of integration. Therefore (22) is certainly less than

$$\frac{\int_1^m [\cosh^{-1}(1 + \pi) - \log 2 + 2 \tan^{-1} 2](\xi^2 \pm 2\xi\rho \cos \alpha + \rho^2)^{-1} d\xi}{\int_1^\infty \left(\cosh^{-1} \xi + 2 \tan^{-1} \frac{2\xi}{(\xi^2 - 1)^{1/2}} \right) (\xi^2 \pm 2\xi\rho \cos \alpha + \rho^2)^{-1} d\xi},$$

which, in turn, is less than

$$(23) \quad (\pi - \log 2 - 2 \tan^{-1} 2) \frac{\int_1^m (\xi^2 \pm 2\xi\rho \cos \alpha + \rho^2)^{-1} d\xi}{M \int_1^\infty (\xi^2 \pm 2\xi\rho \cos \alpha + \rho^2)^{-1} d\xi},$$

† A somewhat elaborate investigation of the character of the function $\Theta(1, \xi)$ for small values of ξ enabled the author to compute (20) and (21) to a higher degree of accuracy than any practical considerations would warrant. See the author's thesis, 1930, in the Library of the University of Wisconsin.

where M is the lower bound of

$$\cosh^{-1} \xi + 2 \tan^{-1} \frac{2\xi}{(\xi^2 - 1)^{1/2}}$$

in the interval $1 \leq \xi \leq \infty$. Moreover, $(\xi^2 \pm 2\xi\rho \cos \alpha + \rho^2)^{-1}$ is always positive, and it is clear that the quotient of the integrals in (23) is less than unity. Therefore (23) is less than

$$\frac{\pi - \log 2 - 2 \tan^{-1} 2}{M} = \frac{\pi - \log 2 - 2 \tan^{-1} 2}{\cosh^{-1} \xi + 2 \tan^{-1} \frac{2\xi}{(\xi^2 - 1)^{1/2}} \Big|_{\xi=1}} = .0748.$$

This result is quite significant, since it states that, even allowing such crude approximations as were made above in estimating the relative error, the latter is always less than 7.48 per cent. This rough estimate of the degree of approximation gives ample justification for the development of the approximate formulas for $\Phi_2(\rho, \alpha)$ by considering† $m = 1$.

Replacing m by 1 in the formulas (20) and (21) and substituting the numerical values of a_1 , a_2 , and a_3 , leads without difficulty to the following expressions. For $\rho \leq 1$,

$$(24) \quad \Phi_2(\rho, \alpha) = \frac{d^2}{\pi} \left\{ - .209 \tan^{-1} \frac{2\rho \sin \alpha}{1 - \rho^2} + 1.703 \sum_{n=1}^{\infty} \frac{\sin (2n-1)\alpha}{(2n-1)^2} \rho^{2n-1} \right. \\ \left. + 1.272 \sum_{n=1}^{\infty} \frac{\sin (2n-1)\alpha}{(2n-1)^3} \rho^{2n-1} \right\},$$

and for $\rho \geq 1$,

$$(25) \quad \Phi_2(\rho, \alpha) = \frac{d^2}{\pi} \left\{ \log^2 2\rho + \pi\alpha - \alpha^2 + .209 \tan^{-1} \frac{2\rho \sin \alpha}{\rho^2 - 1} \right. \\ + 1.703 \sum_{n=1}^{\infty} \frac{\sin (2n-1)\alpha}{(2n-1)^2} \rho^{-2n+1} \\ - 1.272 \sum_{n=1}^{\infty} \frac{\sin (2n-1)\alpha}{(2n-1)^3} \rho^{-2n+1} \\ \left. + 1.282 \log 2\rho - 2.07 \right\}.$$

† The uniformity of the results obtained by using this approximation suggests that the actual error is in the neighborhood of one per cent.

V. GRAPHIC EXPRESSION OF THE RESULTS

If we use the notation of §III, and introduce a new function defined by the equation

$$\Psi = \Phi - \frac{1}{2}(x^2 + y^2),$$

we find that Ψ satisfies

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + 2 = 0,$$

subject to the boundary condition $\Psi = 0$ on the sides of the polygon.† One can easily obtain expressions for the components of the shearing stress in terms of Ψ . They are

$$X_z = G\tau \frac{\partial \Psi}{\partial y},$$

$$Y_z = -G\tau \frac{\partial \Psi}{\partial x},$$

where τ is the angle of twist per unit length, and G is the modulus of rigidity. The tangential traction at any point of the area of the polygon is directed along the tangent to the curve of the family

$$(26) \quad \Psi(x, y) = \text{const.}$$

which passes through the point in question.‡

It will be recalled that in the case of the T-section

$$\Phi = \Phi_1 + \Phi_2 = \frac{x^2 - y^2}{2} + yd + \Phi_2,$$

and the expression for Ψ becomes

$$(27) \quad \Psi = \Phi_2 + yd - y^2.$$

Equation (27) served for computing the level lines of the shearing stress function. A sufficiently large number of these lines is shown in the accompanying drawing of the contour elevations. The width of the web and flange was taken to be unity for convenience.

It may be shown§ that the torsional rigidity of the section is equal to

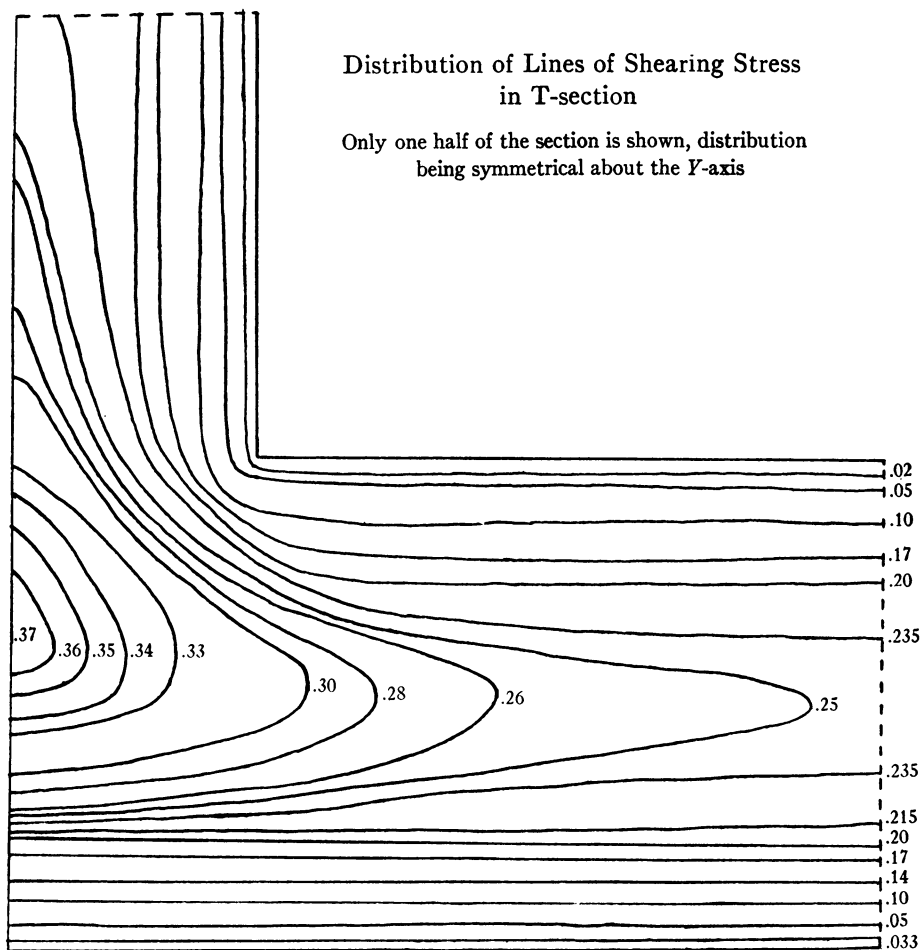
$$C = 2G \iint \Psi dx dy,$$

† Love, loc. cit., pp. 325–327. Trayer and March, loc. cit., pp. 10–13.

‡ See preceding footnote.

§ Trayer and March, loc. cit., p. 12. Love, loc. cit., pp. 325–327.

where G is the modulus of rigidity, and where the integration is carried over the area of the section. In other words, the torsional rigidity of the prism is equal to twice the product of the modulus of rigidity and the volume contained between the surface $z = \Psi(x, y)$ and the plane $z = 0$. From the knowledge



of the contour elevations, the "torsion constant" $K \equiv C/G$ depending solely on the shape and the dimensions of the cross section may be easily computed.

Trayer and March[†] succeeded in developing a set of formulas for the torsion constants of sections whose components are rectangles, by combining results obtained from soap-film tests with known mathematical facts. The value of the torsion constant computed from their formula agrees closely with that obtained from the accompanying graph of the contour elevations.

[†] Loc. cit., p. 12.

VI. CONCLUDING REMARKS

It is necessary to point out that the method of solution of Laplace's equation outlined and illustrated above depends neither upon the number of reentrant angles, nor the special form of the boundary conditions.

Such ingenious devices as were used by Bromwich,[†] Kötter,[‡] and Trefftz[§] do not furnish direct means for solving an important group of classical problems, the solution of which is regarded to be of considerable technical value.

It remains my pleasant duty to acknowledge that Professor H. W. March is responsible for directing my attention to this problem and for offering many helpful suggestions. To Professors R. E. Langer and Warren Weaver my thanks are due for their valuable criticisms.

[†] T. J. P.A. Bromwich, *Proceedings of the London Mathematical Society*, (2), vol. 30 (1930), pp. 165-173.

[‡] Loc. cit.

[§] Loc. cit.